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Decay of Solutions to the Mixed Problem for the Linearized Boltzmann Equation with an External Potential in a Bounded Domain

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Abstract We consider the mixed problem for the linearized Boltzmann equation with a sufficiently smooth external-force potential in a bounded domain whose boundary is a 2-dimensional piecewise linear manifold. We impose the perfectly reflective boundary condition. We do not impose the convexity of the domain. The mixed problem has a unique solution decaying exponentially in time.

§1 Introduction

The nonlinear Boltzmann equation with an external-force potential has the form,

$$(1.1) \qquad f_t + (\xi \cdot \nabla_x - \nabla_x \phi \cdot \nabla_\xi)f = Q(f,f).$$

This equation describes the time evolution of rarefied gas acted upon by the external force $\mathbb{F} = -\nabla\phi$, $\phi = \phi(x)$. $f = f(t,x,\xi)$ is an unknown function denoting the density of the gas particles at the time $t \geqq 0$, at the position $x \in \Omega$, and with the velocity $\xi \in \mathbb{R}^3$, where Ω is a bounded domain $\subset \mathbb{R}^3$. We assume that the gas particles are confined in Ω and are reflected perfectly from the boundary $\partial\Omega$. $Q(\cdot,\cdot)$ denotes the nonlinear collision operator.

We accept the assumption of the cut-off hard potentials. We linearize (1.1) around absolute Maxwellian state. By substituting $f = M+M^{1/2}u$, $M \equiv \exp(-\phi(x)-|\xi|^2/2)$, in (1.1), and by dropping the non-linear term, we obtain,

$$(1.2) \qquad u_t = Bu, \ B \equiv A + (\exp(-\phi))K,$$

where $A \equiv -(\xi \cdot \nabla_x - \nabla_x \phi \cdot \nabla_\xi) + (\exp(-\phi))(-\nu)$. $\nu = \nu(\xi)$ is a multiplication operator. K is an integral operator. ν and K satisfy the following (see [1-2]):

Lemma 1.1. (i) There exist positive constants $\nu_\pm$ such that

$$\nu_- \leqq \nu(\xi) \leqq \nu_+(1+|\xi|).$$

(ii) K is a self-adjoint compact operator in $L^2(\mathbb{R}^3)$.

(iii) $(-\nu+K)$ is a nonpositive operator in $L^2(\mathbb{R}^3)$ whose null space is spanned by $\phi_j \equiv \xi_j\exp(-|\xi|^2/4)$, $j = 1,2,3$, $\phi_4 \equiv \exp(-|\xi|^2/4)$ and $\phi_5 \equiv |\xi|^2\exp(-|\xi|^2/4)$.

We consider the mixed problem for (1.2) with the perfectly reflective boundary condition. We will demonstrate that if $\phi = \phi(x)$ is sufficiently smooth and if $\partial\Omega$ is a 2-dimensional piecewise linear manifold, then the mixed problem has a unique solution decaying exponentially in time. Our main result is Theorem 2.4.

§ 2 The main theorem

We impose the following on Ω:

Assumption 2.1. (i) Ω is a bounded domain of $\mathbb{R}^3$.
(ii) $\partial\Omega$ is a 2-dimensional piecewise linear manifold.

We denote the set of all points of the edges of $\partial\Omega$ by $E(\partial\Omega)$. By $n = n(x)$ we denote the outer unit normal of $\partial\Omega$ at $x \in F(\partial\Omega) \equiv \partial\Omega \setminus E(\partial\Omega)$.

We impose the following on $\phi = \phi(x)$:

Assumption 2.2. (i) $\phi = \phi(x)$ is sufficiently smooth in Ω.
(ii) $\partial^2\phi(x)/\partial x_i \partial x_j$, $i,j = 1,2$, are uniformly bounded in Ω.
(iii) $n(x)\cdot\nabla\phi(x) = 0$, for any $x \in F(\partial\Omega)$.

We consider our problem in $L^2(\Omega\times\mathbb{R}^3)$. Write $\|\cdot\|$ as the norm. By $D(L)$ we denote the domain of an operator L. We define $D(\Lambda) \equiv \{v = v(x, \xi) \in L^2(\Omega\times\mathbb{R}^3);\ \Lambda v \in L^2(\Omega\times\mathbb{R}^3).\ v = v(x, \xi)$ satisfies the perfectly reflective boundary condition,

$$\text{(PRBC)} \qquad (\gamma_+ v(\cdot,\cdot))(x, \xi) = (\gamma_- v(\cdot,\cdot))(x, \xi - 2(n(x)\cdot\xi)n(x)),$$

for any $(x, \xi) \in F_+\}$, where $\gamma_\pm$ denote the trace operators along the characteristic curves of Λ onto $F_\pm \equiv \{(x, \xi) \in F(\partial\Omega)\times\mathbb{R}^3;\ \pm n(x)\cdot\xi > 0\}$. The characteristic curves of Λ are defined by the following system of equations:

$$\text{(SE)} \qquad dx/dt = \xi, \quad d\xi/dt = -\nabla\phi(x).$$

We similarly define $D(A) \equiv \{v = v(x, \xi) \in L^2(\Omega\times\mathbb{R}^3);\ Av \in L^2(\Omega\times\mathbb{R}^3)$, and $v = v(x, \xi)$ satisfies (PRBC) for any $(x, \xi) \in F_+\}$. Since $e^{-\phi(x)}K$ is bounded operator in $L^2(\Omega\times\mathbb{R}^3)$, we can define $D(B) \equiv D(A)$.

Lemma 2.3. The intersection of $\{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu \geq 0\}$ and the point spectrum of B is equal to $\{0\}$. The null space is spanned by $e^{-E(x,\xi)/2}$ and

$E(x, \xi)e^{-E(x,\xi)/2}$.

Consider the mixed problem,

$$u_t = Bu,\ \ t > 0,\ \ u|_{t=0} = u_0 \in L_{\perp}^2(\Omega \times \mathbb{R}^3),$$

where $L_{\perp}^2(\Omega \times \mathbb{R}^3)$ denotes the set of all functions $\in L^2(\Omega \times \mathbb{R}^3)$ which are perpendicular to the null space of B. The following is the main theorem:

Theorem 2.4. The mixed problem has a unique solution $u = u(t)$, which satisfies that there exists positive constants c_{2j}, $j = 1,2$, such that for any $t \geqq 0$,

$$\| u(t) \| \leqq c_{2.1} \| u_0 \| \exp(-c_{2.2}t).$$

The reason why we impose Assumption 2.1,(ii) and Assumption 2.2,(iii).

We do not assume that the domain is convex. Because of this, we cannot appropriately get rid of gas particles which follow the boundary surface, i.e.,we cannot remove characteristic curves of the operator Λ which follow the boundary surface. In the same way as in [4-5], we will set up the resolvent equations as follows:

$$(\mu - B)^{-1} = (\mu - A)^{-1} + (\mu - A)^{-1}\{1 - e^{-\phi}K(\mu - A)^{-1}\}^{-1} e^{-\phi}K(\mu - A)^{-1}.$$

It is nearly impossible to demonstrate that $e^{-\phi}K(\mu - A)^{-1}$ is a compact operator. Hence, in the same way as in [4-5], we will prove, with the aid of Assumption 2.1,(ii) and Assumption 2.2,(iii), that the 4-th power of this operator is compact.

REFERENCES

[1] H. Grad. Asymptotic theory of the Boltzmann equation, II. In "Rarefied Gas Dynamics," (J. A. Laurmann Ed.), Academic Press,N. Y., 26-59, (1963).

[2] H. Grad. Asymptotic equivalence of the Navier-Stokes and nonlinear Boltzmann equations. Proc. Symp. in Appl. Math., 17, 154-183, (1965).

[3] Y. Shizuta-K. Asano. Global solutions of the Boltzmann equation in a bounded convex domain. Proc. Japan. Acad., Ser. A, 53, 3-5, (1977).

[4] M. Tabata. Decay of solutions to the mixed problem with the periodicity boundary condition for the linearized Boltzmann equation with conservative external force. Comm. in Partial Differential Equations, 18, No. 11, 1823-1846, (1993).

[5] M. Tabata. Decay of solutions to the Cauchy problem for the linearized Boltzmann equation with an unbounded external-force potential. TTSP, Vol.23, 6, 741-780, (1994).

[6] K. Asano. On the global solutions of the initial boundary value problem for the Boltzmann equation with an external force. TTSP, 16, 735-761, (1987).

[7] S. Ukai. On the existence of global solutions of the mixed problem for the nonlinear Boltzmann equation. Proc. Japan Acad., Ser. A, 50, 179-184, (1974).